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DIFFUSION IN A TWO-REGION SLAB WITH AN ERODING BOUNDARY

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Abstract—The transient mass diffusion problem with linear irreversible trapping and external sources for a two-region slab which has one of its outer boundaries eroding at a prescribed rate and subjected to boundary condition of the third kind at both boundaries is analyzed and a zeroth order solution is presented for the concentration distribution in the medium as a function of time and position. To illustrate the application, numerical results are presented for the concentration distribution in the medium.

NOMENCLATURE

$\bar{B}$,	function defined in equation (33);
C,	atomic concentration;
C_∞,	ambient concentration;
D,	mass diffusion coefficient;
f,	initial distribution of concentration;
F,	transformed initial distribution;
$\bar{F}$,	integral transform of F;
h_1,	mass transfer coefficient;
H,	dimensionless parameter;
K,	kernel, equation (23);
K_m,	dimensionless parameter;
L,	initial thickness of entire medium;
L_1,	thickness of region 1;
L_2,	$= L - vt$;
N,	normalization integral, equation (42);
S,	external source of implanted atoms;
t,	time variable;
t_1,	$= (L - L_1)/v$;
v,	erosion rate;
w_m,	parameter defined after equation (42);
W,	dependent variable, equation (8);
$\bar{W}$,	integral transform of W;
x,	space variable;
Z,	function defined in equation (15);
$\bar{Z}$,	integral transform of Z.

Greek symbols

α, β,	coefficients in the linear transformation, equation (8);
γ,	eigenvalue;
Γ_m,	dimensionless parameter;
δ_m,	dimensionless parameter;
λ,	trapping rate;
ξ_m,	dimensionless parameter;
ψ,	eigenfunction.

Subscripts

i,	region index (1 or 2);
m,	summation index;
n,	summation index.

INTRODUCTION

PROBLEMS of heat and mass diffusion in composite media with a moving outer boundary is of interest in certain branches of engineering. In high temperature diffusion studies, for example, where thermal evaporation may lead to erosion of the solid surface, the diffusion profiles within the solid are influenced by the movement of the boundary surface [1]. In controlled thermonuclear reactors, particle bombardment on the plasma side of the reactor vessel can cause erosion of the wall [2].

In diffusion problems of this type one of the boundaries is moving at a prescribed rate. In most applications, although the rate of surface erosion is very small, the diffusion rate as well as the thickness involved are also very small. Therefore, the boundary motion, though small, influences the diffusion in the medium. The exact analysis of problems belonging to this type of moving boundary problems has been very limited and is restricted to idealized situations [3]. Recently, related mass diffusion problems involving a moving boundary have been studied [4, 5] by using the method of analysis developed in ref. [6]. Here, we consider such mass diffusion problem for a two layer slab with an external source and a moving boundary, and develop an approximate analytical solution for the determination of the concentration of species as a function of time and position in the medium.

ANALYSIS

Consider a composite medium consisting of two parallel layers of homogeneous slabs in which one of the outer boundaries is moving at a prescribed rate as a result of surface erosion. The erosion rate and the mass diffusion coefficients in both regions are assumed constant. The diffusion, in general, may be influenced by the distribution of sinks or trapping sites in the medium. The trapping sites are assumed inexhaustible and uniformly distributed over each region. The trapping reaction is assumed to be a first order process and irreversible.

Let L be the initial thickness of the entire medium, L_1 the thickness of region 1 and v the erosion rate of the outer surface of region 2. Then the equations governing the diffusion process in such a medium can be written as

$$D_1 \frac{\partial^2 C_1}{\partial x^2}(x,t) - \lambda_1 C_1(x,t) + S_1(x,t) = \frac{\partial C_1}{\partial t}(x,t) \quad \text{in } 0 < x < L_1,\ t > 0, \quad (1)$$

$$D_2 \frac{\partial^2 C_2}{\partial x^2}(x,t) - \lambda_2 C_2(x,t) + S_2(x,t) = \frac{\partial C_2}{\partial t}(x,t)$$

in $L_1 < x < L_2(t) = L - vt,\ 0 < t < t_1 = \dfrac{L - L_1}{v}$. (2)

Clearly, for $\lambda_1 = \lambda_2 = \lambda$ the problem reduces to that of diffusion of a radioactive species with a decay constant given by λ in a two region slab with a moving boundary.

The boundary conditions for the problem are taken as

$$-D_1 \frac{\partial C_1}{\partial x}(x,t) + h_1 C_1(x,t) = h_1 C_{1\infty}, \text{ at } x = 0,\ t > 0 \quad (3)$$

$$C_1(x,t) = C_2(x,t) \quad (4)$$

$$D_1 \frac{\partial C_1}{\partial x}(x,t) = D_2 \frac{\partial C_2}{\partial x}(x,t) \quad (5)$$

(4) and (5) at the interface $x = L_1,\ t > 0$

$$C_2(x,t) = C_{2\infty}, \text{ at } x = L_2(t) = L - vt,\ t > 0 \quad (6)$$

and the initial conditions as

$$C_i(x,t) = f_i(x) \text{ for } t = 0, \quad i = 1 \text{ or } 2. \quad (7)$$

In order to make the boundary conditions for this problem homogeneous, new dependent variables $W_i(x, t)$ are defined as

$$C_i(x,t) = W_i(x,t) + \alpha_i(t)x + \beta_i(t),\ i = 1 \text{ or } 2 \quad (8)$$

where the coefficients $\alpha_i(t)$ and $\beta_i(t)$ are to be so chosen as to make the boundary conditions for the transformed problem homogeneous. The resulting transformed system becomes

$$D_i \frac{\partial^2 W_i}{\partial x^2}(x,t) - \lambda_i W_i(x,t) + Z_i(x,t) = \frac{\partial W_i}{\partial t}(x,t),\ i = 1 \text{ or } 2 \quad (9)$$

subject to the boundary conditions

$$-D_1 \frac{\partial W_1}{\partial x}(x,t) + h_1 W_1(x,t) = 0, \text{ at } x = 0,\ t > 0, \quad (10)$$

$$W_1(L_1, t) = W_2(L_1, t),\ t > 0, \quad (11)$$

$$D_1 \frac{\partial W_1}{\partial x}(x,t)\Big|_{x=L_1} = D_2 \frac{\partial W_2}{\partial x}(x,t)\Big|_{x=L_1},\ t > 0, \quad (12)$$

$$W_2(x,t) = 0, \text{ at } x = L_2(t) = L - vt,\ t > 0, \quad (13)$$

and the initial conditions

$$W_i(x,t) = F_i(x), \text{ for } t = 0,\ i = 1 \text{ or } 2 \quad (14)$$

where

$$Z_i(x,t) = S_i(x,t) - \lambda_i[\alpha_i(t)x + \beta_i(t)] - \frac{d\alpha_i(t)}{dt}x - \frac{d\beta_i(t)}{dt},\quad i = 1 \text{ or } 2, \quad (15)$$

$$F_i(x) = f_i(x) - \alpha_i(0)x - \beta_i(0),\ i = 1 \text{ or } 2 \quad (16)$$

and the coefficients are given by

$$\alpha_1(t) = \frac{D_2 h_1 (C_{2\infty} - C_{1\infty})}{D_1 D_2 + h_1\{D_2 L_1 + D_1[L_2(t) - L_1]\}}, \quad (17)$$

$$\alpha_2(t) = \frac{D_1 h_1 (C_{2\infty} - C_{1\infty})}{D_1 D_2 + h_1\{D_2 L_1 + D_1[L_2(t) - L_1]\}}, \quad (18)$$

$$\beta_1(t) = \frac{D_1 D_2 (C_{2\infty} - C_{1\infty})}{D_1 D_2 + h_1\{D_2 L_1 + D_1[L_2(t) - L_1]\}} + C_{1\infty}, \quad (19)$$

$$\beta_2(t) = \frac{D_1 h_1 L_2(t)(C_{1\infty} - C_{2\infty})}{D_1 D_2 + h_1\{D_2 L_1 + D_1[L_2(t) - L_1]\}} + C_{2\infty}. \quad (20)$$

This problem defined by equations (9)–(20) is now solved using the integral transform technique as described below.

We define the integral transform pair as

$$W_i(x,t) = \sum_{m=1}^{\infty} K_{im}(x,t)\bar{W}_m(t),\quad i = 1 \text{ or } 2, \quad (21)$$

$$\bar{W}_m(t) = \sum_{i=1}^{2} \int_{L_i} K_{im}(x,t)\, W_i(x,t)\,dx \quad (22)$$

where

$$K_{im}(x,t) = \frac{\psi_{im}(x,t)}{N_m(t)^{1/2}} \quad (23)$$

where $\psi_{im}(x, t)$ and $N_m(t)$ ($i = 1$ or 2 and $m = 1, 2, 3, \ldots$) are the eigenfunctions and the normalization integral respectively associated with the eigenvalue problem given by

$$D_i \frac{d^2\psi_i}{dx^2}(x, \gamma) - \lambda_i \psi_i(x, \gamma) + \gamma(t)\psi_i(x, \gamma) = 0, \quad i = 1 \text{ or } 2 \qquad (24)$$

subject to the boundary conditions

$$-D_1 \frac{d\psi_1}{dx}(x, \gamma) + h_1 \psi_1(x, \gamma) = 0, \text{ at } x = 0 \qquad (25)$$

$$\psi_1(L_1, \gamma) = \psi_2(L_1, \gamma) \qquad (26)$$

$$D_1 \frac{d\psi_1}{dx}(x, \gamma)\bigg|_{x=L_1} = D_2 \frac{d\psi_2}{dx}(x, \gamma)\bigg|_{x=L_1} \qquad (27)$$

$$\psi_2(x, \gamma) = 0, \text{ at } x = L_2(t) = L - vt \qquad (28)$$

Transformation of the set of equations (9)–(20) by the application of the integral transform given in equation (22) yields the following infinite set of ordinary differential equations for $\bar{W}_m(t)$:

$$\frac{d\bar{W}_m(t)}{dt} + \gamma_m(t)\bar{W}_m(t) + \sum_{n=1}^{\infty} \bar{W}_n(t)\bar{B}_{mn}(t) = \bar{Z}_m(t) \qquad (29)$$

subject to the initial condition

$$\bar{W}_m(0) = \bar{F}_m(0) \qquad (30)$$

where

$$\bar{Z}_m(t) = \sum_{i=1}^{2} \int_{L_i} K_{im}(x, t) Z_i(x, t) dx \qquad (31)$$

$$\bar{F}_m(0) = \sum_{i=1}^{2} \int_{L_i} K_{im}(x, 0) F_i(x) dx \qquad (32)$$

and

$$\bar{B}_{mn}(t) = \sum_{i=1}^{2} \int_{L_i} K_{im}(x, t) \frac{\partial K_{in}}{\partial t}(x, t) dx \qquad (33)$$

In practice, only a finite number of such equations needs to be considered; the zeroth order solution is obtained by neglecting the summation term in equation (29) entirely. Now equations (29)–(33) are solved for $\bar{W}_m(t)$ using the zeroth order analysis, the resulting expression for $\bar{W}_m(t)$ is inverted using the inversion formula (21) and the linear transformation (8) is applied. The resulting zeroth order solutions for the concentration distributions $C_1(x, t)$ and $C_2(x, t)$ in the regions 1 and 2 are determined as

$$C_1(x, t) = \sum_{m=1}^{\infty} K_{1m}(x, t) \left\{ \bar{F}_m(0) \exp\left[-\int_0^t \gamma_m(t')dt' \right] + \int_0^t \bar{Z}_m(t') \exp\left[-\int_{t'}^t \gamma_m(t'')dt'' \right] dt' \right\} + \frac{D_2(C_{2\infty} - C_{1\infty})(h_1 x + D_1)}{D_1 D_2 + h_1\{D_2 L_1 + D_1[L_2(t) - L_1]\}} + C_{1\infty}, \quad \text{in } 0 \le x \le L_1, t > 0 \qquad (34)$$

$$C_2(x, t) = \sum_{m=1}^{\infty} K_{2m}(x, t) \left\{ \bar{F}_m(0) \exp\left[-\int_0^t \gamma_m(t')dt' \right] + \int_0^t \bar{Z}_m(t') \exp\left[-\int_{t'}^t \gamma_m(t'')dt'' \right] dt' \right\} + \frac{D_1 h_1 (C_{1\infty} - C_{2\infty})[L_2(t) - x]}{D_1 D_2 + h_1\{D_2 L_1 + D_1[L_2(t) - L_1]\}} + C_{2\infty}, \quad \text{in } L_1 \le x \le L_2(t), t > 0 \qquad (35)$$

where

$$\bar{F}_m(0) = \int_0^{L_1} K_{1m}(x, 0) F_1(x) dx + \int_{L_1}^{L} K_{2m}(x, 0) F_2(x) dx \qquad (36)$$

and

$$\bar{Z}_m(t) = \int_0^{L_1} K_{1m}(x, t) Z_1(x, t) dx + \int_{L_1}^{L_2(t)} K_{2m}(x, t) Z_2(x, t) dx \qquad (37)$$

Here, the form of the normalized eigenfunctions $K_{im}(x, t)$, ($i = 1$ or 2), of the eigenvalue problem defined by equations (24)–(28) will depend on the permissible values for the eigenvalues $\gamma_m(t)$. It can be shown for all m (see the Appendix for proof) that

$$\gamma_m(t) > \min\{\lambda_1, \lambda_2\} \qquad (38)$$

Equation (38) implies that the eigenfunctions of equations (24) for the region of smallest trapping rate will be of the form of sines and cosines whereas for the other region, the solution may be of the form of hyperbolic and/or trigonometric series. Equation (38) is therefore a mathematical restriction for the eigenfunctions spanning the space of solutions for the region of smallest trapping rate. Included in this general result and assuming $\lambda_1 < \lambda_2$, we have the following possibilities:

$$\lambda_1 < \gamma_m(t) < \lambda_2 \quad (39)$$

$$\lambda_1 < \lambda_2 < \lambda_m(t) \quad (40)$$

as well as the very special case of

$$\gamma_m(t) = \lambda_2 \text{ for certain values of } m \text{ and } t. \quad (41)$$

Clearly the above situation applies for systems where the rate at which the diffusing atoms are trapped to either defects or the lattice atoms of region 2 is bigger than that of region 1.

Based on the above possibilities for the spectrum of eigenvalues, $\gamma_m(t)$, the permissible eigenfunctions for regions 1 and 2 can be determined. When the eigenfunctions, $\psi_{im}(x, t)$, are known, the normalization integral, $N_m(t)$, for the whole region is determined by

$$N_m(t) = \int_0^{L_1} \psi_{1m}^2(x, t)\mathrm{d}x + \int_{L_1}^{L_2(t)} \psi_{2m}^2(x, t)\mathrm{d}x \quad (42)$$

If we define

$$w_{1m} = (\gamma_m - \lambda_1)^{1/2}; \quad w_{2m} = (|\gamma_m - \lambda_2|)^{1/2}$$

$$\Gamma_{1m} = \frac{w_{1m}L_1}{D_1^{1/2}}; \quad \Gamma_{2m} = \frac{w_{2m}L_2}{D_2^{1/2}}$$

$$H = \frac{h_1 L_1}{D_1}; \quad K_m = \left(\frac{D_1}{D_2}\right)\frac{w_{1m}}{w_{2m}}$$

$$\delta_{1m} = K_m\left(\cos\Gamma_{1m} - \frac{\Gamma_{1m}}{H}\sin\Gamma_{1m}\right)$$

$$\delta_{2m} = \sin\Gamma_{1m} + \frac{\Gamma_{1m}}{H}\cos\Gamma_{1m}$$

$$\xi_m = 2\Gamma_{2m}\left(1 - \frac{L_1}{L_2}\right)$$

then the permissible eigenfunctions and corresponding values of the integrals appearing in equation (42), for regions 1 and 2, are given below.

Region 1

The eigenfunction $\psi_{1m}(x, t)$ is given by

$$\psi_{1m}(x, t) = \sin\left(\frac{w_{1m}x}{D_1^{1/2}}\right) + \frac{\Gamma_{1m}}{H}\cos\left(\frac{w_{1m}x}{D_1^{1/2}}\right) \quad (43)$$

and the integral term is evaluated as

$$\int_0^{L_1} \psi_{1m}^2(x, t)\mathrm{d}x = \frac{L_1}{2}\left\{\left(\frac{\Gamma_{1m}}{H}\right)^2 + 1 + \left[\left(\frac{\Gamma_{1m}}{H}\right)^2 - 1\right]\frac{\sin 2\Gamma_{1m}}{2\Gamma_{1m}} + \frac{2}{H}\sin^2\Gamma_{1m}\right\} \quad (44)$$

Region 2

For this region, the three cases to be considered according to equations (38)–(41) are as follows:

(a) *For* $\lambda_1 < \gamma_m(t) < \lambda_2$. The eigenfunction $\psi_{2m}(x, t)$ is given by

$$\psi_{2m}(x, t) = \delta_{1m}\sinh\left[\frac{w_{2m}(x - L_1)}{\sqrt{D_2}}\right] + \delta_{2m}\cosh\left[\frac{w_{2m}(x - L_1)}{\sqrt{D_2}}\right] \quad (45)$$

and the integral term is evaluated as

$$\int_{L_1}^{L_2(t)} \psi_{2m}^2(x, t)\mathrm{d}x = \frac{(\delta_{2m}^2 - \delta_{1m}^2)}{2} \times [L_2(t) - L_1]\left[1 - \frac{\sinh\xi_m}{\xi_m}\right] \quad (46)$$

where $\gamma_m(t)$ are the solutions of the transcendental equation

$$\delta_{2m}\cosh\left[\Gamma_{2m}\left(1 - \frac{L_1}{L_2}\right)\right] + \delta_{1m}\sinh\left[\Gamma_{2m}\left(1 - \frac{L_1}{L_2}\right)\right] = 0. \quad (47)$$

(b) For $\lambda_1 < \lambda_2 < \gamma_m(t)$.

$$\psi_{2m}(x, t) = \delta_{1m}\sin\left[\frac{w_{2m}(x - L_1)}{\sqrt{D_2}}\right] + \delta_{2m}\cos\left[\frac{w_{2m}(x - L_1)}{\sqrt{D_2}}\right] \quad (48)$$

and

$$\int_{L_1}^{L_2(t)} \psi_{2m}^2(x, t)\mathrm{d}x = \left(\frac{\delta_{2m}^2 + \delta_{1m}^2}{2}\right) \times [L_2(t) - L_1]\left[1 - \frac{\sin\xi_m}{\xi_m}\right] \quad (49)$$

where $\gamma_m(t)$ are the roots of

$$\delta_{2m}\cos\left[\Gamma_{2m}\left(1 - \frac{L_1}{L_2}\right)\right] + \delta_{1m}\sin\left[\Gamma_{2m}\left(1 - \frac{L_1}{L_2}\right)\right] = 0. \quad (50)$$

(c) *The special case of* $\gamma_m(t) = \lambda_2$. This may occur only for a specific value of t; for this particular time, the eigenfunction $\psi_{2m}(x, t)$ takes the form

$$\psi_{2m}(x, t) = \frac{D_1}{D_2}\frac{\Gamma_{1m}}{L_1}\left(\cos\Gamma_{1m} - \frac{\Gamma_{1m}}{H}\sin\Gamma_{1m}\right) \times (x - L_1) + \delta_{2m} \quad (51)$$

and the integral term becomes

$$\int_{L_1}^{L_2(t)} \psi_{2m}^2(x, t)\,dx = \left(\frac{D_1}{D_2}\frac{\Gamma_{1m}}{L_1}\right)^2 \frac{[L_2(t) - L_1]^3}{3} \left(\cos\Gamma_{1m} - \frac{\Gamma_{1m}}{H}\sin\Gamma_{1m}\right)^2. \quad (52)$$

The particular time t at which these relations are valid is determined from

$$[L_2(t) - L_1]\frac{D_1}{D_2}\frac{\Gamma_{1m}}{L_1}\left(\cos\Gamma_{1m} - \frac{\Gamma_{1m}}{H}\sin\Gamma_{1m}\right) + \left(\sin\Gamma_{1m} + \frac{\Gamma_{1m}}{H}\cos\Gamma_{1m}\right) = 0. \quad (53)$$

For the case when $\lambda_1 = \lambda_2 = \lambda$, it follows from the proof in the Appendix that for all m,

$$\gamma_m(t) > \lambda$$

and the solutions are then given by equations (43), (44) and (48)–(50) with λ_i, $i = 1$ or 2, replaced by λ. It should be mentioned that in the limit as the erosion velocity approaches zero (stationary boundary), the zeroth order solution given by equations (34)–(37) becomes the exact solution for the problem.

The accuracy of the zeroth order solution is expected to decrease with increasing erosion velocity v as illustrated for the case of one region problem [4]. In that reference, an estimate of the error introduced by neglecting the summation in equation (29) is made possible by comparing the zeroth order solution with the exact solution available for that case. In addition, if the problem involves sources strongly dependent on position and peaking near the moving boundary, the convergence of the series being slower at the boundary, the accuracy of the solution near the moving boundary may not be good.

RESULTS AND DISCUSSION

To illustrate the application of the foregoing analysis, we consider a two-region slab subjected to homogeneous boundary conditions of first and third kind at the moving and stationary boundaries respectively. The initial concentrations are zero in both regions; for times $t > 0$ the medium is subjected to a delta function source distribution in the region 2 given by

$$S_2(x, t) = S_0\delta(x - x_0), \quad (S_1(x, t) = 0)$$

In this case $\alpha_i(t)$, $\beta_i(t)$ and $f_i(x)$, $i = 1$ or 2, are all zero as a consequence of the choice for the boundary and initial conditions. Then the solutions described by equations (34)–(37) reduce to

$$C_i(x, t) = S_0 \sum_{m=1}^{\infty} K_{im}(x, t) \int_0^t K_{2m}(x_0, t')\exp\left[-\int_{t'}^{t} \gamma_m(t'')\,dt''\right]dt', \; i = 1 \text{ or } 2 \quad (54)$$

where

$$K_{im}(x, t) = \frac{\psi_{im}(x, t)}{\sqrt{N_m(t)}}$$

and $\psi_{im}(x, t)$, $\gamma_m(t)$ and $N_m(t)$ are as defined previously.

Tables 1 and 2 show the concentrations $C_i(x, t)$, $i = 1$ or 2, calculated from the solutions (54) for two different values of the diffusion coefficient D_2 ($D_2 = 10^{-8}\,cm^2\,sec^{-1}$ and $10^{-9}\,cm^2\,sec^{-1}$ respectively). The

Table 1. Values of $C(x, t)$ in a two region slab with $D_1 = 10^{-6}\,cm^2\,s^{-1}$ and $D_2 = 10^{-8}\,cm^2\,s^{-1}$

x(cm)	Region 1 $C_1(\times 10^{-18}\,cm^{-3})$	x(cm)	Region 2 $C_2(\times 10^{-18}\,cm^{-3})$
0.0	0.1685	1.0	1.131
0.1	0.2534	1.009	2.037
0.2	0.3483	1.018	2.951
0.3	0.4446	1.027	3.878
0.4	0.5427	1.036	4.820
0.5	0.6404	1.045	5.780
0.6	0.7378	1.054	6.764
0.7	0.8349	1.063	7.765
0.8	0.9325	1.072	8.858
0.9	1.031	1.081	9.813
1.0	1.131	1.0855	11.56
		1.0889	4.402
		1.0900	0.0

Table 2. Values of $C(x, t)$ in a two region slab with $D_1 = 10^{-6}\,cm^2\,s^{-1}$ and $D_2 = 10^{-9}\,cm^2\,s^{-1}$

x(cm)	Region 1 $C_1(\times 10^{-18}\,cm^{-3})$	x(cm)	Region 2 $C_2(\times 10^{-18}\,cm^{-3})$
0.0	0.2797	1.0	1.396
0.1	0.4109	1.009	6.530
0.2	0.5920	1.018	12.02
0.3	0.7694	1.027	17.95
0.4	0.8994	1.036	25.03
0.5	1.005	1.045	33.45
0.6	1.086	1.054	43.59
0.7	1.166	1.063	55.88
0.8	1.251	1.072	70.70
0.9	1.330	1.081	88.52
1.0	1.396	1.0885	97.90
		1.0889	90.71
		1.0900	0.0

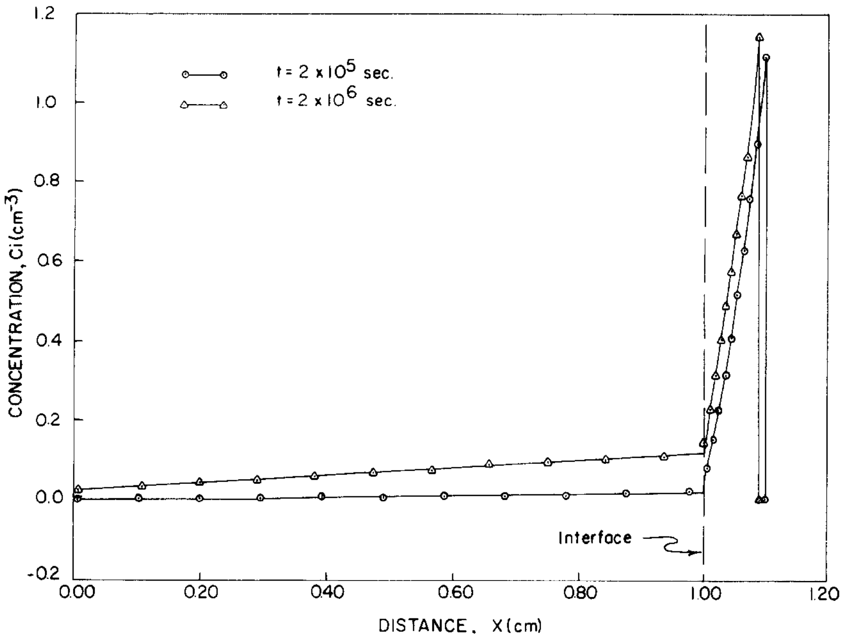

FIG. 1. Concentration profiles at two different times in a two region slab with a moving boundary.

other parameters, which are common for both tables, are chosen as

$$L = 1.1\,\text{cm}; L_1 = 1.0\,\text{cm}; D_1 = 10^{-6}\,\text{cm}^2\,\text{sec}^{-1};$$
$$\lambda_1 = 5 \times 10^{-8}\,\text{sec}^{-1};$$
$$\lambda_2 = 5 \times 10^{-7}\,\text{sec}^{-1}; h_1 = 5 \times 10^{-6}\,\text{cm}\,\text{sec}^{-1};$$
$$v = 5 \times 10^{-9}\,\text{cm}\,\text{sec}^{-1};$$
$$t = 2 \times 10^6\,\text{sec}; S_0 = 10^{14}\,\text{cm}^{-2}\,\text{sec}^{-1}\text{and}$$
$$x_0 = x_0(t) = 0.999\,L - vt$$

The spatial distributions of the concentrations for the parameters of Table 1 and for two different times are illustrated in Fig. 1.

In computing the integrals of $\gamma_m(t)$ appearing in equation (54), we fit the eigenvalues satisfied by the transcendental equations (47) and (50) to a polynomial of fifth degree in t (with linear coefficients) by linear regression. This procedure significantly reduced the computation time since the resulting integrals could be performed analytically. The other integrals appearing in equation (54) that could not be performed analytically were performed numerically by the Gaussian quadrature scheme using 64 points of quadrature.

In the example considered here all the eigenvalues, $\gamma_m(t)$, were bigger than both λ_1 and λ_2 (case b). However by modifying the input parameters (e.g. $L = 2.0\,\text{cm}$) some eigenvalues can be found in the interval (λ_1, λ_2). In addition, as D_2 becomes smaller compared to D_1, the difference between two adjacent eigenvalues decreases and more terms in the summation are needed for good convergence of the solution.

For example, to produce the results in Table 1, 80 terms in the series were sufficient whereas in Table 2, 160 terms were needed to obtain the same degree of convergence.

The accuracy of the results presented in Tables 1 and 2 cannot be ascertained since no rigorous solution for this problem exists. However, for the case presented in Table 2, since $D_1 \gg D_2$, it is possible to approximate this problem by a one-region slab consisting of only the second region. We have performed calculations for this equivalent one-region problem. The results agreed with the values presented in Table 2 to within 10% for all interior points in region (2). This gives confidence in the method presented in the paper since the accuracy of the solutions to one-region problems have been validated in ref. [4]. Furthermore, this comparison shows that the first 160 eigenvalues for the two region problem were calculated with sufficient accuracy, without missing any.

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APPENDIX

The proof of relation (38)

We write equations (24) for regions 1 and 2

$$D_1 \frac{d^2\psi_{1m}}{dx^2} - \lambda_1\psi_{1m} + \gamma_m\psi_{1m} = 0,$$

$$D_2 \frac{d^2\psi_{2m}}{dx^2} - \lambda_2\psi_{2m} + \gamma_m\psi_{2m} = 0.$$

The first equation is operated on with the operator

$$\int_0^{L_1} \psi_{1m}\,dx$$

and the second with the operator

$$\int_{L_1}^{L_2} \psi_{2m}\,dx.$$

The results are added, the integral terms involving the second derivatives are integrated by parts once and the boundary conditions (25)–(28) are applied.

We obtain

$$(\gamma_m - \lambda_1)\int_0^{L_1} \psi_{1m}^2\,dx + (\gamma_m - \lambda_2) \times \int_{L_1}^{L_2} \psi_{2m}^2\,dx = h_1\,\psi_{1m}^2\Big|_{x=0} + D_1 \int_0^{L_1} \times \left(\frac{d\psi_{1m}}{dx}\right)^2 dx + D_2 \int_{L_1}^{L_2} \left(\frac{d\psi_{2m}}{dx}\right)^2 dx > 0$$

since all the quantities in the RHS of the equality are positive. Then we write

$$(\gamma_m - \lambda_1)\int_0^{L_1} \psi_{1m}^2\,dx + (\gamma_m - \lambda_2)\int_{L_1}^{L_2} \psi_{2m}^2\,dx > 0$$

or

$$\gamma_m(t) > \min\{\lambda_1, \lambda_2\}$$

which is the result given by equation (38).

DIFFUSION DANS UNE PLAQUE A DEUX COUCHES AVEC UNE FRONTIERE QUI S'ERODE

Résumé—On étudie le problème transitoire de diffusion, massique avec un piégeage irreversible et linéaire et des sources externes pour une plaque à deux couches dont chacune à sa face externe qui s'érode à une vitesse donnée et qui est soumise à une condition aux limites de troisième espèce. On recherche la solution d'ordre zéro pour la concentration dans le milieu en fonction du temps et de la position. Pour illustrer son application, des résultats numériques sont présentés pour la distribution de concentration dans le milieu.

DIFFUSION IN EINER ZWEISCHICHTIGEN WAND MIT EINER ERODIERENDEN BERANDUNG

Zusammenfassung—Das instationäre Stofftransportproblem mit linear irreversiblem 'trapping' und äußeren Quellen wird für eine zweischichtige Wand, deren eine äußere Berandung mit einer vorgegebenen Geschwindigkeit erodiert und an deren beiden äußeren Berandungen Randbedingungen dritter Art herrschen, untersucht und eine Lösung nullter Ordnung für die Konzentrationsverteilung im Medium als Funktion von Ort und Zeit angegeben. Um die Anwendung zu erläutern, werden zahlenmäßige Ergebnisse der Konzentrationsverteilung in dem Medium mitgeteilt.

ДИФФУЗИЯ В ДВУХСЛОЙНОМ СТЕРЖНЕ С КОРРОДИРУЮЩЕЙ ГРАНИЦЕЙ

Аннотация — С учетом линейного необратимого поглощения и внешних источников анализируется нестационарная диффузия массы в стержне, одна из внешних границ которого корродирует с заданной интенсивностью, а на двух других имеет место граничное условие третьего рода. Дано решение в нулевом приближении для распределения концентрации в среде в зависимости от времени и координаты и представлены численные результаты.